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A note on the one-particle and Hilbert–Schmidt cohomologies of the Poincaré group in three spacetime dimensions

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Abstract. Cocycles for $P_+^\uparrow(2+1)$ with values in a Hilbert space are studied. It is found that cocycles for irreducible representations (apart from the case of vanishing momentum) are trivial. Further, cocycles for the action $V(\cdot)V^{-1}$ with values in the representation space of V are investigated, and found to be trivial. Some consequences for physics are obtained.

1. Introduction

This article is part of a series which presents the solution and use in mathematical physics of group cohomology [1-5].

Group cohomology arises from problems of group covariance in non-Fock representations of the canonical commutation relations (CCR) and of the canonical anti-commutation relations (CAR).

For the CAR, one studies unitary implementability, of a symmetry group G of the one-particle space $\mathscr{H}$, in representations defined by pure quasifree gauge invariant states ω_P, where P is a projection in $\mathscr{H}$. The necessary and sufficient condition for this to occur is

$$V_g P V_g^{-1} - P \in B(\mathscr{H})_2 \qquad \text{(the Hilbert–Schmidt operators on } \mathscr{H}\text{)}$$

for each $g \in \mathrm{G}$. The state ω_P is a Fock state if and only if $P \in B(\mathscr{H})_2$. V_g is the representation of G in $\mathscr{H}$.

The case of the CCR falls into two parts. First come the displaced Fock representations over a complex pre-Hilbert space τ, whose completion is $\mathscr{H}$. These are defined by linear functionals, F, on τ. The condition of unitary implementability, provided that any G-invariant F vanishes on τ, is then given by the necessary and sufficient condition

$$V_g F - F \in \mathscr{H} \qquad \text{for each } g \in \mathrm{G}.$$

Second come the symplectically transformed states. Starting from the Fock representation of the Weyl operators, $\{W(f): f \in \mathscr{H}\}$ we may obtain a non-Fock representation (those which do not contain a vacuum state) $W_T(f) = W(Tf)$ where T is a symplectic operator on $\mathscr{H}$. The operator T is regarded as a real-linear operator in the complex space $\mathscr{H}$. One can show (see [4]) that $T = U \exp A$ where U is a unitary operator in $\mathscr{H}$ and A is an anti-linear operator in $\mathscr{H}$. W_T is equivalent to W if and only if $A \in B(\mathscr{H})_2$. The necessary and sufficient condition that the group G be unitarily implementable in W_T is given as

$$V_g A V_g^{-1} - A \in B(\mathscr{H})_2 \qquad \text{for each } g \in \mathrm{G}.$$

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All these three problems fall into the domain of 1-cocycles of G with values in a Hilbert space. For details of the derivation of the conditions, the reader can consult [1, 2, 4] for the CCR, and [3] for the CAR.

Having explained why 1-cohomology arises, we may begin the process of solution.

2. Cocycles

A 1-cocycle for the connected Lie group G, which has a strongly continuous unitary representation V in a Hilbert space $\mathscr{H}$, is a continuous map ψ: $\mathrm{G} \to \mathscr{H}$ which satisfies the condition

$$\psi(gh) = V_g \psi(h) + \psi(g) \qquad \text{for } g, h \in \mathrm{G}.$$

The cocycle ψ: $\mathrm{G} \to \mathscr{H}$ is a *coboundary* if there is a vector $\xi \in \mathscr{H}$ such that

$$\psi(g) = V_g \xi - \xi$$

for all $g \in \mathrm{G}$. Two cocycles ψ_1 and ψ_2 are called *cohomologous* (or *equivalent*) cocycles if

$$\psi_1(g) - \psi_2(g) = V_g \xi - \xi$$

for some fixed vector $\xi \in \mathscr{H}$, i.e. if the difference $\psi_1 - \psi_2$ is a coboundary.

The set of cocycles is written $Z^1(\mathrm{G}, \mathscr{H})$ and the set of coboundaries is written $B^1(\mathrm{G}, \mathscr{H})$. The group of 1-cohomology classes is defined by

$$H^1(\mathrm{G}, \mathscr{H}) = Z^1(\mathrm{G}, \mathscr{H}) / B^1(\mathrm{G}, \mathscr{H}).$$

A cocycle for G which is analytic at the identity has values in $\mathscr{H}_\omega$, the dense set of analytic vectors for the group representation in $\mathscr{H}$. The set of these analytic cocycles is denoted by $Z^1_\omega(\mathrm{G}, \mathscr{H}_\omega)$ and the analytic coboundaries by $B^1_\omega(\mathrm{G}, \mathscr{H}_\omega) = B^1(\mathrm{G}, \mathscr{H}) \cap Z^1_\omega(\mathrm{G}, \mathscr{H}_\omega)$. The corresponding cohomology group is

$$H^1_\omega(\mathrm{G}, \mathscr{H}_\omega) = Z^1_\omega(\mathrm{G}, \mathscr{H}_\omega) / B^1_\omega(\mathrm{G}, \mathscr{H}_\omega).$$

It is proved in [6] that if $\psi \in Z^1(\mathrm{G}, \mathscr{H})$, then there is a $\psi' \in Z^1_\omega(\mathrm{G}, \mathscr{H}_\omega)$ such that $\psi - \psi' \in B^1(\mathrm{G}, \mathscr{H})$. Moreover, we have [6]

$$H^1(\mathrm{G}, \mathscr{H}) \simeq H^1_\omega(\mathrm{G}, \mathscr{H}_\omega).$$

We can define similar structures for the Lie algebra, $\boldsymbol{G}$, of G. If V is the representation of G, then π is the representation of $\boldsymbol{G}$ obtained by using the formula

$$\mathrm{i}\pi(X)\xi = \frac{\mathrm{d}}{\mathrm{d}t} V(\mathrm{e}^{tX})\xi \bigg|_{t=0}$$

for $X \in \boldsymbol{G}$ and $\xi \in \mathscr{H}_\omega$. The derivative is the *strong* derivative in the topology of $\mathscr{H}$.

A cocycle for $\boldsymbol{G}$ with values in $\mathscr{H}_\omega$ is a linear map η: $\boldsymbol{G} \to \mathscr{H}_\omega$ satisfying

$$\eta([X, Y]) = \pi(X)\eta(X) - \pi(Y)\eta(X)$$

for $X, Y \in \boldsymbol{G}$. $[X, Y]$ is the Lie bracket of X and Y. η is a *coboundary* if there is a $\xi \in \mathscr{H}_\omega$ such that

$$\eta(X) = \pi(X)\xi \qquad \text{for each } X \in \boldsymbol{G}.$$

The set of cocycles of G is written as $Z^1(G, \mathcal{H}_\omega)$ and the set of coboundaries by $B^1(G, \mathcal{H}_\omega)$. The cohomology group is defined as before

$$H^1(G, \mathcal{H}_\omega) = Z^1(G, \mathcal{H}_\omega)/B^1(G, \mathcal{H}_\omega).$$

Let us define, as in [6], the map Δ: $Z^1_\omega(\mathrm{G}, \mathcal{H}_\omega) \to Z^1(G, \mathcal{H}_\omega)$ by

$$(\Delta\psi)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathrm{e}^{tX})\bigg|_{t=0}$$

Again, the derivative is the strong derivative in the topology of $\mathcal{H}$.

Δ is an isomorphism between $B^1_\omega(\mathrm{G}, \mathcal{H}_\omega)$ and $B^1(G, \mathcal{H}_\omega)$. Also, Δ maps $H^1(\mathrm{G}, \mathcal{H}_\omega)$ injectively into $H^1(G, \mathcal{H}_\omega)$. The mapping is an isomorphism when G is simply connected.

It is shown in [7] that if G contains an Abelian normal subgroup, N, then any $\psi \in Z^1(\mathrm{G}, \mathcal{H})$ can be written as

$$\psi(g) = \psi_F(g) + \psi_1(g)$$

where ψ_1 has values in those vectors of $\mathcal{H}$ which are invariant under the action of N, and ψ_F is of the form

$$\psi_F(g) = M_g F - F$$

for each $g \in \mathrm{G}$. Here, $F \in \overline{D^+(\mathrm{N})}$, a space which contains $\mathcal{H}$ as a dense (non-closed!) subspace, and M_g is a representation of G which agrees with V_g on $\mathcal{H}$ (this is a reformulation of theorem 7.3 of [7]).

We shall call any $\psi \in Z^1(\mathrm{G}, \mathcal{H})$ of the form

$$\psi(g) = V_g F - F$$

with F not necessarily in $\mathcal{H}$, a *quasi-coboundary* (this terminology was first introduced by Falkowski [10]). Notice that we abuse mathematical nicety by calling M_g by the name V_g. This does not disturb the results which we obtain.

We denote the set of quasi-coboundaries by $Z^1_Q(\mathrm{G}, \mathcal{H})$. Clearly, $B^1(\mathrm{G}, \mathcal{H}) \subset Z^1_Q(\mathrm{G}, \mathcal{H})$. The corresponding cohomology group is defined by

$$H^1_Q(\mathrm{G}, \mathcal{H}) = Z^1_Q(\mathrm{G}, \mathcal{H})/B^1(\mathrm{G}, \mathcal{H}).$$

It is easy to show that $H^1_Q(\mathrm{G}, \mathcal{H}) \simeq H^1_Q(\mathrm{G}, \mathcal{H}_\omega)$. Here,

$$H^1_Q(\mathrm{G}, \mathcal{H}_\omega) = Z^1_Q(\mathrm{G}, \mathcal{H}_\omega)/B^1(\mathrm{G}, \mathcal{H}_\omega)$$

and $Z^1_Q(\mathrm{G}, \mathcal{H}_\omega)$ consists of elements from $Z^1_Q(\mathrm{G}, \mathcal{H}) \cap Z^1_\omega(\mathrm{G}, \mathcal{H}_\omega)$.

Quasi-coboundaries also arise for the Lie algebra of a Lie group. Indeed, η: $G \to \mathcal{H}_\omega$ is a quasi-coboundary for the Lie algebra if there exists F, with $F \notin \mathcal{H}$ in general, such that

$$\eta(X) = \pi(X)F.$$

Strictly speaking, we should ask for the condition that $f \to F(\pi(X)f)$ define a continuous linear functional on $\mathcal{H}_\omega$, but nothing is really lost by writing the original condition, and the results obtained are the same.

We write $Z^1_Q(G, \mathcal{H}_\omega)$, $B^1_\omega(G, \mathcal{H}_\omega)$ and $H^1_Q(G, \mathcal{H}_\omega)$ for the quasi-coboundaries, coboundaries and the cohomology classes of the quasi-coboundaries, respectively. We have the relation

$$H^1_Q(G, \mathcal{H}_\omega) = Z^1_Q(G, \mathcal{H}_\omega)/B^1_\omega(G, \mathcal{H}_\omega).$$

Furthermore, one has the relation $H^1_Q(\mathrm{G}, \mathscr{H}_\omega) \simeq H^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$. To prove this, we need the following supporting result.

Lemma 1 [2]. Suppose $t \to V_t$ is a representation of $\mathbb{R}$ in a space $\mathscr{M}$ which contains a Hilbert space $\mathscr{H}$ as a dense subspace, and that V_t is unitary when restricted to $\mathscr{H}$. Then the following two conditions are equivalent:

(1) $F \in \mathscr{M}$ satisfies $V_t F - F \in \mathscr{H} \qquad \forall t \in \mathbb{R}$.

(2) $F \in \mathscr{M}$ satisfies $V_t F - F \in \mathscr{H} \qquad \forall t \in (-\varepsilon, \varepsilon)$

for any $\varepsilon > 0$.

Theorem 1. $H^1_Q(\mathrm{G}, \mathscr{H}_\omega) \simeq H^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$.

Proof. If $\psi \in Z^1_Q(\mathrm{G}, \mathscr{H}_\omega)$ then $\Delta\psi \in Z^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$ and the map Δ respects classes of cocycles. Hence to each $\psi \in Z^1_Q(\mathrm{G}, \mathscr{H}_\omega)$ there corresponds a $\eta \in Z^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$.

Now let $\eta \in Z^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$. We have

$$\eta(X) = \mathrm{i}\pi(X)F \qquad \text{for some } F,\ \forall X \in \boldsymbol{G}.$$

Furthermore, $i^n \pi(X)^n F \in \mathscr{H}_\omega$ $\forall n \geq 1$ and there is an $\varepsilon > 0$ with

$$\sum_{n=1}^{\infty} \frac{\mathrm{i}^n t^n}{n!} \pi(X)^n F \qquad (0 \leq t < \varepsilon)$$

convergent. Thus $[\exp(\mathrm{i}t\pi(X) - 1]F \in \mathscr{H}_\omega$ $\forall t \in (-\varepsilon, \varepsilon)$. Write $V_t = \exp(\mathrm{i}t\pi(X))$, and apply lemma 1, so that we obtain

$$\exp(\mathrm{i}t\pi(X))F - F \in \mathscr{H}_\omega \qquad \forall t \in \mathbb{R} \qquad \forall X \in \boldsymbol{G}.$$

If $U(\exp(tX)) = \exp(\mathrm{i}t\pi(X))$, then $U(\mathrm{e}^{tX})F - F \in \mathscr{H}_\omega$ for all one-parameter groups in G.

If $g \in \mathrm{G}$, we can write

$$g = \mathrm{e}^{X_1} \ldots \mathrm{e}^{X_n}$$

for some $X_1, \ldots, X_n \in \boldsymbol{G}$. Suppose $g = \mathrm{e}^{X_1}\mathrm{e}^{X_2}$, then we have

$$\begin{aligned} U(g)F - F &= U(\mathrm{e}^{X_1}\mathrm{e}^{X_2})F - F \\ &= U(\mathrm{e}^{X_1})[U(\mathrm{e}^{X_2})F - F] + U(\mathrm{e}^{X_1})F - F \in \mathscr{H}_\omega. \end{aligned}$$

The general case follows by induction. Hence, to each $\eta \in Z^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$ there corresponds a $\psi \in Z^1_Q(\mathrm{G}, \mathscr{H}_\omega)$. This demonstrates an isomorphism between $Z^1_Q(\mathrm{G}, \mathscr{H}_\omega)$ and $Z^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$ and consequently we obtain

$$H^1_Q(\mathrm{G}, \mathscr{H}_\omega) \simeq H^1_Q(\boldsymbol{G}, \mathscr{H}_\omega)$$

and this proves the theorem.

Proposition 1. (*a*) Let G_1 and G_2 be two connected Lie groups such that $p: \mathrm{G}_1 \to \mathrm{G}_2$ is a continuous surjective homomorphism. If V is a strongly continuous unitary representation of G_2 on a Hilbert space $\mathscr{H}$, then each $\psi_2 \in Z^1(\mathrm{G}_2, \mathscr{H})$ defines an element $\psi_1 \in Z^1(\mathrm{G}_1, \mathscr{H})$ in the representation $U = V \circ p$.

If p is an isomorphism, then we obtain

$$Z^1(\mathrm{G}_1, \mathscr{H}) \simeq Z^1(\mathrm{G}_2, \mathscr{H}).$$

(*b*) If V and U are unitarily equivalent strongly continuous unitary representations of the connected Lie group G, in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, then the resulting cohomologies are isomorphic.

Proof. (*a*) Define $\psi_1(h) = \psi_2(g)$ where $g = p(h)$ with $h \in G_1$ and $g \in G_2$. Then we have

$$U_h = V_g$$

and

$$\psi_2(g_1 g_2) = V_{g_1}\psi_2(g_2) + \psi_2(g_1)$$

implies the relation

$$\psi_1(h_1 h_2) = U_{h_1}\psi_1(h_2) + \psi_1(h_1).$$

This proves the first part. If p is an isomorphism, $V = U \circ p^{-1}$ and the same argument works to prove the isomorphism of the cocycle spaces.

(*b*) If S is the unitary operator such that

$$V_g = SU_g S^{-1}$$

and if ψ is a cocycle for U, the cocycle ψ_S defined by

$$\psi_S(g) = S\psi(g)$$

is a cocycle for V. This establishes the result, and the proposition.

If the connected Lie group G contains a compact Lie group K, then each cocycle ψ of G has an equivalent cocycle ψ' such that $\psi'(k) = 0$ $\forall k \in$ K. It is possible to combine this property with analyticity at the identity of G [1].

Let ${}^K H^1_Q(\mathrm{G}, \mathcal{H}_\omega)$ be the cohomology classes of cocycles with values in $\mathcal{H}_\omega$ and which vanish on K and let ${}^K H^1_Q(\boldsymbol{G}, \mathcal{H}_\omega)$ be the cohomology classes of cocycles η: $\boldsymbol{G} \to \mathcal{H}_\omega$ with $\eta(X_K) = 0$ for the Lie algebra elements X_K of $\boldsymbol{K}$, then we have the following proposition.

Proposition 2. $H^1_Q(\mathrm{G}, \mathcal{H}_\omega) \simeq {}^K H^1_Q(\mathrm{G}, \mathcal{H}_\omega) \simeq {}^K H^1_Q(\boldsymbol{G}, \mathcal{H}_\omega)$. The proof is a combination of the remarks made and theorem 1.

A useful result is the following.

Lemma 2. Suppose that G = [G, G], i.e. G is equal to its commutator subgroup. Then any cocycle of G, for the trivial representation, is identically zero.

Proof. We have $\psi(gh) = \psi(g) + \psi(h)$, from the cocycle condition. Hence $\psi(g) = -\psi(g^{-1})$.

Since G = [G, G] we can assume, for each $g \in$ G, $\exists k, h \in$ G such that $g = khk^{-1}h^{-1}$. Then

$$\psi(g) = \psi(k) + \psi(h) + \psi(k^{-1}) + \psi(h^{-1}) = 0.$$

This proves the lemma.

There are many examples of such G. One is SU(1, 1), which is the group we consider here.

3. Cocycles for SU(1, 1)

We consider now quasi-coboundaries for unitary irreducible representations of SU(1, 1).

All the unitary irreducible representations of SU(1, 1) are listed in the very rich paper of Bargmann [8]. The group has three parameters, and the generators are $\mathcal{J}_0$ for the compact subgroup, and $\mathcal{J}_1$ and $\mathcal{J}_2$ for the other two one-parameter subgroups. If F is such that for any irreducible representation V of SU(1, 1) with

$$V_g F - F \in \mathcal{H}$$

for $g \in \mathrm{SU}(1,1)$, and where $\mathcal{H}$ is the representation space, and if $V_g F - F \in Z^1_Q(\mathrm{SU}(1,1), \mathcal{H}_\omega)$ and vanishes on the compact subgroup of SU(1, 1), then F satisfies

$$\mathcal{J}_0 F = 0 \qquad \text{and} \qquad \mathcal{J}_1 F \in \mathcal{H}_\omega, \qquad \mathcal{J}_2 F \in \mathcal{H}_\omega.$$

For such an F, we obtain, by direct calculation, that

$$\|F\|^2 \leq 4\{\|\mathcal{J}_1 F\|^2 + \|\mathcal{J}_2 F\|^2\}.$$

The calculations are not difficult, but somewhat lengthy, and so we do not present them here.

This implies that ${}^K H^1_Q(\mathrm{su}(1,1), \mathcal{H}_\omega) = \{0\}$, where su(1, 1) is the Lie algebra of the group SU(1, 1). This, in turn, gives us the result

$$H^1_Q(\mathrm{SU}(1,1), \mathcal{H}) = \{0\}.$$

Now suppose that V is any representation of SU(1, 1), which may or may not contain the identity representation. If ψ is a cocycle for V, then we have

$$\psi(g) = \int_\Omega^\oplus \psi(g, \omega)\, \mathrm{d}\alpha(\omega)$$

where Ω is the Borel space for the direct integral decomposition of the representation V

$$V = \int_\Omega^\oplus V^\omega\, \mathrm{d}\alpha(\omega).$$

Each $\psi(\cdot, \omega)$ is (for almost all $\omega \in \Omega$) a cocycle for V^ω. The proof of this can be found in [9].

If ψ is a quasi-coboundary for V, then we may assume that each $\psi(\cdot, \omega)$ is also a quasi-coboundary, for each $\omega \in \Omega$ (we may neglect sets of measure zero, and we take advantage of this liberty). Furthermore, we assume ψ to be analytic at the identity and to vanish on the compact subgroup of SU(1, 1), from which we obtain the same properties for each $\psi(\cdot, \omega)$ in the direct integral decomposition. From lemma 2, we deduce that no mention need be made of the trivial representation, as SU(1, 1) is equal to its commutator group, whence all the corresponding cocycles are identically zero.

Let us represent the elements of the Lie algebra su(1, 1) by X, in the infinitesimal representation of V, and let us represent the same element by X^ω in the infinitesimal representation of V^ω. Assuming

$$\psi \in Z^1_Q(\mathrm{SU}(1,1), \mathcal{H}_\omega)$$

and $\psi(k) = 0$ for elements k of the compact subgroup, then we obtain

$$XF = \int_\Omega^\oplus X^\omega F^\omega\, \mathrm{d}\alpha(\omega)$$

where we have

$$V_g F - F = \int_\Omega^\oplus (V_g^\omega F^\omega - F^\omega)\, \mathrm{d}\alpha(\omega).$$

Since we assume $XF \in \mathscr{H}$, then we have

$$\|XF\|^2 = \int_\Omega \|X^\omega F^\omega\|^2 \, \mathrm{d}\alpha(\omega) < \infty.$$

We have assumed $\mathscr{J}_0 F = 0$, so $\mathscr{J}_0^\omega F^\omega = 0$, and hence we obtain, using our earlier result for irreducible representations,

$$\|F^\omega\|^2 \leq 4\{\|\mathscr{J}_1^\omega F^\omega\|^2 + \|\mathscr{J}_2^\omega F^\omega\|^2\}$$

and this means that

$$\int_\Omega \|F^\omega\|^2 \, \mathrm{d}\alpha(\omega) \leq 4 \sum_{l=1}^{2} \int_\Omega \|\mathscr{J}_l^\omega F^\omega\|^2 \, \mathrm{d}\alpha(\omega).$$

Therefore we obtain $F \in \mathscr{H}$. All this can be written as a theorem.

Theorem 2. For any unitary representation of SU(1, 1), the quasi-coboundaries are all true coboundaries, and so $H^1_Q(\mathrm{SU}(1,1), \mathscr{H})$ is trivial. Moreover, $\|F\|^2 \leq 4\{\|\mathscr{J}_1 F\|^2 + \|\mathscr{J}_2 F\|^2\}$ if $\mathscr{J}_0 F = 0$ and $\psi(g) = V_g F - F$ is analytic at the identity.

4. Cocycles for $P_+^\uparrow(2+1)$

Suppose, now, that V is an irreducible representation of $P_+^\uparrow(2+1)$, the Poincaré group of $2+1$ spacetime dimensions. $\mathbb{R}^3$ is an invariant subgroup of $P_+^\uparrow(2+1)$. Indeed,

$$P_+^\uparrow(2+1) = \mathbb{R}^3 \circledS \mathrm{SO}(2,1)$$

where $\circledS$ denotes the semi-direct product. Moreover, there is a continuous surjective homomorphism $p\colon \mathrm{SU}(1,1) \to \mathrm{SO}(2,1)$, so that we obtain a continuous surjective homomorphism $p\colon \mathbb{R}^3 \circledS \mathrm{SU}(1,1) \to \mathbb{R}^3 \circledS \mathrm{SO}(2,1)$. Hence any cocycle of $P_+^\uparrow(2+1)$ can be realised as a cocycle of $\mathbb{R}^3 \circledS \mathrm{SU}(1,1)$. This is a consequence of proposition 1.

Any cocycle of $P_+^\uparrow(2+1)$ can be written as

$$\psi(q) = \psi_F(g) + \psi_1(g)$$

where $\psi_1(g)$ has values in the $\mathbb{R}^3$-invariant vectors of the Hilbert space, and $\psi_F(g)$ is a quasi-coboundary. This is because $\mathbb{R}^3$ is invariant and Abelian in $P_+^\uparrow(2+1)$. If the irreducible representation V corresponds to momentum $p \neq 0$ (i.e. $p^2 > 0$, $p^2 < 0$ or $p^2 = 0$, $p \neq 0$) then $\psi_1(g) \equiv 0$ for all $g \in P_+^\uparrow(2+1)$. Hence, for these representations, the cocycles are all quasi-coboundaries. Therefore let us consider quasi-coboundaries for $P_+^\uparrow(2+1)$. These give us quasi-coboundaries for $\mathbb{R}^3 \circledS \mathrm{SU}(1,1)$. In particular, these are quasi-coboundaries for SU(1, 1), and theorem 2 implies that they must be true coboundaries. (This follows from the domination of the cocycle function F given by

$$\|F\|^2 \leq 4\{\|\mathscr{J}_1 F\|^2 + \|\mathscr{J}_2 F\|^2\}$$

where we have assumed $\mathscr{J}_0 F = 0$.) It follows, then, that all quasi-coboundaries for unitary irreducible representations are true coboundaries. If the unitary irreducible representation does not belong to $p \equiv 0$, then every cocycle, being a quasi-coboundary, is a true coboundary. We have proved the following theorem.

Theorem 3. Let V be any irreducible representation of $P_+^\uparrow(2+1)$. Then

$$H_Q^1(P_+^\uparrow(2+1), \mathscr{H}) = \{0\}.$$

If V does not correspond to vanishing momentum, then

$$H^1(P_+^\uparrow(2+1), \mathscr{H}) = \{0\}.$$

If a unitary representation, V, of $P_+^\uparrow(2+1)$ does not contain the identity representation, then every quasi-coboundary is a true coboundary. Moreover, if also V does not contain the representation of vanishing momentum, then all cocycles are true coboundaries. This follows from using proposition 2 and the calculations leading to the domination of F by the generators $\mathscr{J}_1$ and $\mathscr{J}_2$ of SU(1, 1).

5. Hilbert–Schmidt cohomology for $P_+^\uparrow(2+1)$

Let V be a unitary irreducible representation of $P_+^\uparrow(2+1)$ which does not correspond to vanishing momentum. Then V acts in the Hilbert space

$$\mathscr{K} = L^2(\mathbb{R}^2; \mathbb{C}; \mathrm{d}\mu)$$

where $\mathrm{d}\mu$ is the invariant measure on the appropriate hyperboloid.

Suppose that ψ: $P_+^\uparrow(2+1) \to B(\mathscr{K})_2$ satisfies

$$\psi(gh) = V_g\psi(h)V_g^{-1} + \psi(g)$$

and that ψ is continuous. Then ψ is a $B(\mathscr{K})_2$-valued cocycle. Here ψ may either be a linear or anti-linear operator. It is proved in chapter 4 of [10] that the cocycles ψ are in one-to-one correspondence with

(1) cocycles for $V \otimes V$ with values in $L^2(\mathbb{R}^2; \mathscr{K}; \mathrm{d}\mu)$ if ψ is anti-linear or

(2) cocycles for $V \otimes \bar{V}$ with values in $L^2(\mathbb{R}^2; \mathscr{K}; \mathrm{d}\mu)$ if ψ is linear, where $\bar{V} = CVC$, and C is a conjugation, i.e. $C^2 = \mathbb{1}$ and C is anti-linear in $\mathscr{K}$.

If V does not correspond to vanishing momentum, then $V \otimes V$ and $V \otimes \bar{V}$ can be reduced and expressed as direct integrals of representations of $P_+^\uparrow(2+1)$ which do not correspond to vanishing momentum. This can be done as in [11]. Moreover, the trivial representation of $P_+^\uparrow(2+1)$ does not occur in the decompositions.

From this and the remarks at the end of § 4, it follows that all the cocycles for $V \otimes V$ or $V \otimes \bar{V}$ must be coboundaries.

Theorem 4. Suppose V is an irreducible unitary representation of $P_+^\uparrow(2+1)$ which does not correspond to vanishing momentum. Then all cocycles ψ with

$$\psi: P_+^\uparrow(2+1) \to B(\mathscr{K})_2$$

for the action $V_g(\cdot)V_g^{-1}$, $g \in P_+^\uparrow(2+1)$, whether ψ is linear or anti-linear, are true coboundaries, i.e. there exists $H \in B(\mathscr{K})_2$ with

$$\psi(g) = V_g H V_g^{-1} - H$$

and H is linear or anti-linear, according to the linearity or anti-linearity of ψ.

Theorem 5. A representation of the CCR, which is of displaced Fock type, or a symplectically transformed Fock representation, and which has $P_+^\uparrow(2+1)$ unitarily implemented in Fock space, is itself unitarily equivalent to the Fock representation.

A quasi-free, gauge invariant representation of the CAR which has $P_+^\uparrow(2+1)$ unitarily implemented in Fock space, is itself unitarily equivalent to the Fock representation or to the anti-Fock representation.

Proof. Combine § 1 with theorems 3 and 4.

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